Monte Carlo Simulation of the Midcourse Guidance for Lunar Flights

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A detailed description and numerical results of a Monte Carlo simulation that was set up to determine the fuel requirements and final accuracy of the midcourse phase of the lunar missions is presented. The example considered applies two midcourse corrections to null the errors in three terminal variables. Linear perturbation theory was assumed to be valid. A proposed simulation for missions in which the burnout errors are so large that nonlinear effects are important also is described. The aspects of this simulation which are presented are the basic system for the fuel and accuracy analysis, the technique used for the trajectory computations, the determination of the minimum velocity increment (if applicable), computation of midcourse correction errors, and estimated machine time per run. Cumulative probability functions are computed in two simple cases by both the direct-integration approach and the Monte Carlo method. A discussion of the statistical aspects of the Monte Carlo technique also is included.

I. Introduction

In most of the midcourse guidance analyses that have been performed thus far, the procedure has been to assume a correction logic and then to determine how much fuel is necessary to obtain a high probability of no fuel depletion. An error analysis then is performed to determine the probability of mission success. In both parts of the analysis just described, the final result usually is obtained by the numerical integration of a multivariate probability-density function over a multidimensional space. Computer programs have been set up which rapidly handle integrals involving one to four dimensions. However, in some of the problems that need solution, the dimensionality is six or more. For example, if it is desired to perform an error analysis on a one-correction attitude-controlled spacecraft that corrects errors in three terminal variables, a six-dimensional Gaussian integral is necessary, since the equations that relate the final errors to midcourse errors involve products of Gaussian random variables.‡ When multiple corrections are made or nonlinearities become important, the dimensionality can increase easily above six.

The Monte Carlo technique can be used to circumvent the difficulties involved in integrating over multidimensional spaces. The basic idea involved is that of simulating the midcourse phase of the mission many times using randomly generated samples of the system errors and then performing a statistical analysis of the data obtained. For this technique to give meaningful results, a large number of runs are necessary (of the order of 1000). It is one of the purposes of the paper to show that a large number of runs can be made with a reasonable amount of machine time.

In this paper a detailed description of a Monte Carlo simulation that was set up at Space Technology Laboratories to determine the fuel requirements and final accuracy of the midcourse phase of lunar missions is given. The midcourse phase consisted of two attitude-controlled corrections to null the indicated errors in three controlled variables. How this simulation could be modified to handle in an optimum manner problems where only two terminal variables are controlled is discussed in Appendix C. It was assumed that linear perturbation theory was valid.

A problem of future interest is the simulation of missions in which the burnout errors are so large that nonlinear effects are important. A proposed simulation for handling the nonlinear problem also is discussed. The aspects of this simulation which are presented are the basic system for the fuel and accuracy analysis, the technique used for the trajectory computations, the determination of the minimum velocity increment (if applicable), computation of midcourse correction errors, and estimated machine time per run.

In order to obtain some quantitative feeling as to how accurate the Monte Carlo technique is, cumulative probability functions are computed in two simple cases by both the direct-integration approach and the Monte Carlo method.

II. Monte Carlo Simulation of a Two-Correction Mission

A. Real-Time Correction Logic

In this section a nonoptimum but reasonable real-time logic is described for the execution of the midcourse guidance when two corrections are used to null the errors in three terminal variables. An earth-to-moon flight where $\delta\bar{b} = (\delta b_1, \delta b_2, \delta V_\infty)$§ is nulled (see Fig. 1) is used as an example, but the technique applies equally well to other three-terminal-variable/two-correction missions. It will be apparent also how the technique can be extended to n-correction missions where three terminal variables are controlled.

After injection from an earth parking orbit, the spacecraft is tracked, and a least-squares estimate of the perturbation in the nominal‖ spacecraft state vector is obtained from (see Ref. 1 and Appendix E)

$$\overline{\Delta X} = (A_0{}^T A_0 + \Lambda_{BO}{}^{-1})^{-1}\{(A_0{}^T A_0) \times [(A_0{}^T A_0)^{-1} A_0{}^T \Delta\bar{y}_0] + \Lambda_{BO}{}^{-1}\,\Delta\tilde{\bar{X}}_{BO}\}$$
$$= (A_0{}^T A_0 + \Lambda_{BO}{}^{-1})^{-1}[A_0{}^T\,\Delta\bar{y}_0 + \Lambda_{BO}{}^{-1}\,\Delta\tilde{\bar{X}}_{BO}] \quad (1)$$

where $\Delta\bar{y}_0$ is the column vector

$$\Delta\bar{y}_0 = [(y_{i\text{measured}} - y_{i\text{nominal}})/\sigma_i] \quad (2)$$

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‡ A discussion of the reasons for the necessity of a six-dimensional integral is presented in Appendix A.

§ (b_1, b_2) are coordinates in the impact parameter plane, and V_∞ is the velocity at infinity as would be seen by a moon-based observer and is perpendicular to the impact parameter plane.

‖ "Nominal" referred to here means the state vector that would be obtained if the pre-injection tracking and the injection maneuver were perfect.

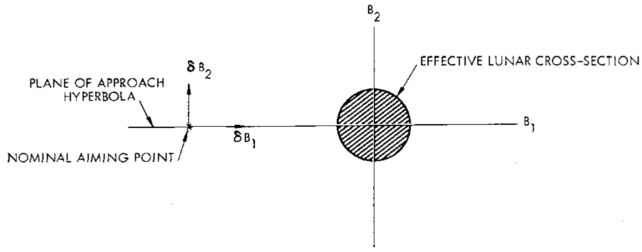

Fig. 1 Impact-parameter plane coordinates

The y_i referred to in (2) is the ith radar observation, and σ_i^2 is the a priori variance of that measurement. The A_0 matrix represents the weighted regression coefficients,

$$A_0 = [(1/\sigma_i)(\partial y_i/\partial x_j)] \tag{3}$$

and Λ_{BO} is the covariance matrix of the error in the state vector caused by the preinjection tracking and the injection maneuver. The partial derivatives of (3) are evaluated with respect to the "nominal" after the programmed maneuver. The best estimate of the error in the maneuver is

$$\Delta\tilde{\bar{X}}_{BO} = 0 \tag{4}$$

and the covariance matrix of the tracking error is

$$\begin{aligned} E[\Delta\tilde{\bar{X}}\Delta\tilde{\bar{X}}^T] &= (A_0^TA_0 + \Lambda_{BO}^{-1})^{-1}\{A_0^TIA_0 + \Lambda_{BO}^{-1}\Lambda_{BO}(\Lambda_{BO}^{-1})^T\}[(A_0^TA_0 + \Lambda_{BO}^{-1})^{-1}]^T \\ &= (A_0^TA_0 + \Lambda_{BO}^{-1})^{-1}(A_0^TA_0 + \Lambda_{BO}^{-1}) \times (A_0^TA_0 + \Lambda_{BO}^{-1})^{-1} \\ &= (A_0^TA_0 + \Lambda_{BO}^{-1})^{-1} \end{aligned} \tag{5}$$

Let $\Phi_{\Delta b}(t_f, t)$ be the matrix that transforms perturbations in the state vector at time t to perturbations in the terminal variables. Then the covariance matrix of uncorrected "miss" in the impact parameter plane is $\Phi_{\Delta b}(t_f, t_0)\Lambda_{BO}\,\Phi_{\Delta b}^T(t_f, t_0)$, and the convariance matrix of tracking uncertainty at time t is $\Phi_{\Delta b}(t_f, t)\ (A_0^TA_0 + \Lambda_{BO}^{-1})^{-1}\ \Phi_{\Delta b}^T(t_f, t)$. Let $B(t)$ be a (3×3) matrix that transforms changes in the velocity coordinates $\Delta\bar{V}_t = (\Delta\dot{x}_t, \Delta\dot{y}_t, \Delta\dot{z}_t)$ to changes in the impact parameter plane coordinates. Therefore, under the logic that all of the indicated error will be removed in a correction, the velocity required is

$$\Delta\bar{V}_t = -B^{-1}(t)\ [\Delta\bar{b}_{BO} + \Delta\bar{b}_{TR}(t)] \tag{6}$$

where $\Delta\bar{b}_{BO}$ is the true error, and $\Delta\bar{b}_{TR}(t)$ is the tracking error. These two components cannot be separated, since only their sum can be observed. However, $\Delta\bar{b}_{TR}(t)$ is decreasing with time because of the "smoothing" effect of the tracking network.

The choice of the time to make the first correction is based on accuracy considerations. The velocity error in the correction can be expressed as follows:

$$\delta\bar{V}_t = (\partial\Delta\bar{V}_t/\partial\theta_t)\ \delta\theta_t + (\partial\Delta\bar{V}_t/\partial\phi_t)\delta\phi_t + k_a\Delta\bar{V}_t + k_b(\Delta\bar{V}_t/\Delta V_t) \tag{7}$$

where $\delta\theta_t$ and $\delta\phi_t$ are attitude orientation errors, k_a is the accelerometer error, and k_b is the thrust shutdown uncertainty. The effect of this error on the terminal variables is $B(t)\delta\bar{V}_t$. The time of the correction should be chosen to satisfy the following conditions:

1) The uncertainty caused by the tracking data should be small compared to the indicated error, i.e., the square roots of the diagonal terms of $\Phi_{\Delta b}(t_f,t)\ (A_0^TA_0 + \Lambda_{BO}^{-1})\ \Phi_{\Delta b}(t_f, t)$ should be small compared to $\Delta\bar{b}_{BO} + \Delta\bar{b}_{TR}(t)$.

2) The miss caused by errors in performing the correction should be no greater than the same order of magnitude as the tracking uncertainty. [The square roots of the diagonal terms of $B(t)E[\delta\bar{V}_t\delta\bar{V}_t^T]B^T(t)$ should be of the same order as the corresponding terms of $\Phi_{\Delta b}(t_f, t)\ (A_0^TA_0 + \Lambda_{BO}^{-1})\ \Phi_{\Delta b}^T(t_f, t)$.] This condition is necessary, since it is possible that the correction errors caused by orientation errors can increase with time because the velocity required to null the indicated error usually increases with time.

Suppose that a time for the first correction has been chosen such that the two constraints just mentioned have been satisfied. The trajectory must be re-established after the correction so that another correction can be made (if necessary). The new a priori matrix of the uncertainty in the state of the spacecraft is computed from

$$\Lambda_{BO_1} = (A_0^TA_0 + \Lambda_{BO}^{-1})^{-1} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Lambda_{\delta V_{t_1}} \end{array}\right] \tag{8}$$

where, from (7),

$$\begin{aligned} \Lambda_{\delta V_{t_1}} &= E(\delta\bar{V}_{t_1}\,\delta\bar{V}_{t_1}^T) \\ &= \left(\frac{\partial\Delta\bar{V}_{t_1}}{\partial\theta_{t_1}}\right)\left(\frac{\partial\Delta\bar{V}_{t_1}}{\partial\theta_{t_1}}\right)^T \sigma_{\theta_{t_1}}{}^2 + \left(\frac{\partial\Delta\bar{V}_{t_1}}{\partial\phi_{t_1}}\right)\left(\frac{\partial\Delta\bar{V}_{t_1}}{\partial\phi_{t_1}}\right)^T \\ &\quad \sigma_{\theta_{t_1}}{}^2 + (\Delta\bar{V}_{t_1})(\Delta\bar{V}_{t_1})^T\left[\sigma_{k_a}{}^2 + \sigma_{k_b}{}^2\frac{1}{(\Delta_{t_1}\bar{V})^2}\right] \end{aligned} \tag{9}$$

From tracking data obtained after the correction, the estimated perturbation from the "nominal" state (i.e., the best estimate of the state before the correction modified by adding the components of a perfect velocity correction to the velocity components of the precorrection state) is computed from

$$\Delta\tilde{\bar{X}} = (A_1^TA + \Lambda_{BO_1}^{-1})^{-1} \times \left\{A_1^T\Delta\bar{y}_1 + (A_0^TA + \Lambda_{BO}^{-1})^{-1}\,\Delta\tilde{\bar{X}}_{TR_1} + \Lambda_{\delta V_{t_1}}\left[\begin{array}{c} 0 \\ \hline \delta\tilde{\bar{V}}_{t_1} \end{array}\right]\right\} \tag{10}$$

In this equation, A_1 is a matrix of regression coefficients computed about the best estimate of the state after the correction. Therefore

$\Delta\tilde{\bar{X}}_{TR_1}$ = best estimate of perturbation from best tracking estimate

= 0

$\delta\tilde{\bar{V}}_{t_1}$ = best estimate of error in correction

= 0

and the covariance matrix of $\Delta\tilde{\bar{X}}$ is

$$E[\Delta\tilde{\bar{X}}\Delta\tilde{\bar{X}}^T] = (A_1^TA_1 + \Lambda_{BO_1}^{-1})^{-1} \tag{11}$$

The problem now remains to choose the time of the second correction. As an aid to deciding when the second correction should be made, the covariance matrix of miss which would result after a correction can be computed by transforming $\Lambda_{\delta V_{t_2}}$ and $E[\Delta\tilde{\bar{X}}\Delta\tilde{\bar{X}}^T]$ to a covariance matrix of miss in the impact parameter plane. The second correction should be made when the miss caused by making the correction (which is composed of correction error and tracking error) is small compared to the indicated error and small enough to insure a sufficiently high probability of success. Or, in more precise terms, making the correction should increase the probability of success.

After the second correction, the a priori uncertainty in the state vector of the spacecraft is

$$\Lambda_{BO_2} = (A_1^TA_1 + \Lambda_{BO_1}^{-1})^{-1} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Lambda_{\delta V_{t_2}} \end{array}\right] \tag{12}$$

and the least squares estimate of the change in the state as determined by additional tracking data is

$$\Delta\tilde{\bar{X}} = (A_2^TA_2 + \Lambda_{BO_2}^{-1})^{-1}\,A_2^T\Delta\bar{y}_2 \tag{13}$$

Table 1 Trajectory characteristics

Launch azimuth	= 91° from Cape Canaveral
100-naut mile parking orbit, circular	
Injection latitude	= −0.322° } in Gulf of Guinea
Injection longitude	= 1.058° } in Gulf of Guinea
Velocity at injection	= 35970.02 fps
Pericynthion altitude	= 70.0 naut miles
Time to pericynthion	= 73.4 hr

B. Technique of Choosing Correction Times for Monte Carlo Simulation

For the purpose of the fuel and error analysis, fixed correction times will be chosen, and all the indicated error will be removed. Although this is not an optimum logic (nor the logic that would be used during an actual flight), it is convenient from the standpoint of facilitating the design of a preliminary Monte Carlo simulation and will give conservative fuel and accuracy estimates. This is true, since a constrained logic must use at least as much fuel and be no more accurate than an unconstrained logic. Holding the correction times fixed allows matrices that do not vary with time to be used as inputs to the Monte Carlo simulation; hence the machine time necessary to perform the analysis with a high degree of confidence is reasonable.

The procedure that actually was used to obtain the numerical results that are presented in this paper will be described. To be specific, considerable detail will be shown for an earth-to-moon flight, but the procedure is similar for moon-to-earth trajectories. The important parameters of the earth-to-moon trajectory that was used as a nominal are shown in Table 1. The assumed covariance matrix of injection errors is shown in Table 2.#

The first step in the procedure involves transforming the injection covariance matrix to impact-parameter plane errors. The equation for this transformation is

$$\Lambda_{\Delta b BO} = \Phi_{\Delta b}(t_f, t_0)\Lambda_{BO}\Phi_{\Delta b}{}^T(t_f, t_0) \tag{14}$$

where $\Phi_{\Delta b}$ (t_f, t_0) is defined in Table 3, and Λ_{BO} is specified in Table 2. The numerical result of (14) is shown in Table 4. After 12 hr of tracking, the uncertainty in the impact-parameter plane coordinates is shown in Table 5. Upon comparing the corresponding diagonal terms of the matrices of Tables 4 and 5, it is seen that 12 hr from injection is a reasonable time to choose as the fixed time of correction, since the standard deviations of the burnout errors are about 10 to 100 times the tracking errors, depending on which terminal coordinate is being compared.

Next, it is necessary to determine the accuracy of the correction. If $B(t_1)$ is a 3×3 matrix relating the components of a velocity correction to changes in impact-parameter-plane coordinates at the time of the first correction, the velocity required can be expressed as

$$\Delta \bar{V}_{t_1} = -B^{-1}(t_1)\ [\Delta\bar{b}_{BO} + \Delta\bar{b}_{TR}(t_1)] \tag{15}$$

and the covariance matrix of $\Delta\bar{V}_{t_1}$ is

$$\Lambda_{\Delta V t_1} = B^{-1}(t_1)\ [\Lambda_{\Delta b BO} + \Lambda_{\Delta b TR_1}][B^{-1}(t_1)]^T \tag{16}$$

Numerical values for $\Lambda_{\Delta b BO}$, $\Lambda_{\Delta b TR_1}$, $B(t_1)$, and $\Lambda_{V t_1}$ are shown in Tables 4–7, respectively. The normalized eigenvectors and the square roots of the corresponding eigenvalues were computed by the 7090, and the result is

$$\sigma_{\dot{x}_1'} = 34.7 \text{ fps} \qquad \bar{e}_1 = (0.8834, -0.3836, -0.2691)$$

$$\sigma_{\dot{y}_1'} = 4.7 \text{ fps} \qquad \bar{e}_2 = (0.4599, 0.6001, 0.6544)$$

$$\sigma_{\dot{z}_1'} = 1.5 \text{ fps} \qquad \bar{e}_3 = (-0.8954, -0.7019, 0.7065)$$

From these results, it can be seen that the direction of $\bar{e}_1$ is "preferred," since its eigenvalue is more than 50 times the next largest eigenvalue.

Table 2 Covariance matrix of earth burnout errors[a]

$(0.400250E04)^2$	−0.294600E05	−0.661500E02	0.657100E01	−0.445100E02	0.275000E02
−0.830902	$(0.885833E01)^2$	0.126400E00	−0.194800E-01	0.132000E-00	−0.815500E-01
−0.799058	0.689882	$(0.206833E\text{-}01)^2$	−0.108200E-04	0.732800E-04	−0.452700E-04
0.101601	−0.136092	−0.032375	$(0.161586E\text{-}01)^2$	−0.115600E-03	−0.745500E-04
−0.568828	0.762214	0.181226	−0.365939	$(0.195499E\text{-}01)^2$	−0.128900E-03
0.435151	−0.583057	−0.138621	−0.292202	−0.417586	$(0.157892E\text{-}01)^2$

$$\begin{bmatrix} \sigma_r^2 & \sigma_{rv} & \sigma_{r\beta} & \sigma_{rA} & \sigma_{r\alpha} & \sigma_{r\delta} \\ \rho_{rv} & \sigma_v^2 & \sigma_{v\beta} & \sigma_{vA} & \sigma_{v\alpha} & \sigma_{v\delta} \\ \rho_{r\beta} & \rho_{v\beta} & \sigma_\beta^2 & \sigma_{\beta A} & \sigma_{\beta\alpha} & \sigma_{\beta\delta} \\ \rho_{rA} & \rho_{vA} & \rho_{\beta A} & \sigma_A^2 & \sigma_{A\alpha} & \sigma_{A\delta} \\ \rho_{r\alpha} & \rho_{v\alpha} & \rho_{\beta\alpha} & \rho_{A\alpha} & \sigma_\alpha^2 & \sigma_{\alpha\delta} \\ \rho_{r\delta} & \rho_{v\delta} & \rho_{\beta\delta} & \rho_{A\delta} & \rho_{\alpha\delta} & \sigma\ ^2 \end{bmatrix}$$

[a] All units are in feet, degrees, and feet per second.

Table 3 Matrix transforming injection errors to impact parameter plane errors $\Phi_{\Delta b}(t_f, t_0)$[a]

−0.6556000E03	−0.7670000E06	−0.3052700E08	−0.9260000E06	0.1059000E07	−0.1673700E08
0.6970000E02	0.2055000E06	0.1115900E08	−0.2991000E07	−0.2233100E08	−0.6232000E07
0.6860000E-02	0.7990000E01	−0.8030000E02	0.3000000E00	0.1810000E02	−0.3520000E02

$$\begin{bmatrix} \frac{\partial b_1}{\partial r} & \frac{\partial b_1}{\partial v} & \frac{\partial b_1}{\partial \beta} & \frac{\partial b_1}{\partial A} & \frac{\partial b_1}{\partial \alpha} & \frac{\partial b_1}{\partial \delta} \\ \frac{\partial b_2}{\partial r} & \frac{\partial b_2}{\partial v} & \frac{\partial b_2}{\partial \beta} & \frac{\partial b_2}{\partial A} & \frac{\partial b_2}{\partial \alpha} & \frac{\partial b_2}{\partial \delta} \\ \frac{\partial V_\infty}{\partial r} & \frac{\partial V_\infty}{\partial v} & \frac{\partial V_\infty}{\partial \beta} & \frac{\partial V_\infty}{\partial A} & \frac{\partial V_\infty}{\partial \beta} & \frac{\partial V_\infty}{\partial \delta} \end{bmatrix}$$

[a] All units are in feet, degrees, and feet per second.

In Table 2 and the following tables presenting covariance matrices, it was decided that, since the matrices are symmetric, the lower triangle would be used to indicate the correlation coefficients of their corresponding elements.

Table 4 Covariance matrix of uncorrected impact parameter plane error $\Lambda_{\Delta b_{BO}}$

$$\begin{bmatrix} (0.504477E07)^2 & -0.736116E13 & -0.249422E09 \\ -0.955921 & (0.152645E07)^2 & 0.700632E08 \\ -0.987421 & 0.916676 & (0.500716E02)^2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{b_1}^2 & \sigma_{b_1b_2} & \sigma_{b_1V_\infty} \\ \rho_{b_1b_2} & \sigma_{b_2}^2 & \sigma_{b_2V_\infty} \\ \rho_{b_1V_\infty} & \rho_{b_2V_\infty} & \sigma_{V_\infty 2} \end{bmatrix}$$

The difficulty in determining the accuracy of the correction lies in the non-Gaussian statistics of the error in the correction, as can be seen from (7), which is repeated here:

$$\delta\bar{V}_{t_1} = (\partial\bar{V}_{t_1}/\partial\theta_{t_1})\delta\theta_{t_1} + (\partial\bar{V}_{t_1}/\partial\phi_{t_1})\delta\phi_{t_1} + k_a\Delta\bar{V}_{t_1} + k_b(\Delta\bar{V}_{t_1}/|\Delta\bar{V}_{t_1}|) \quad (17)$$

Note that this expression involves products of random variables. To circumvent this difficulty, a "typical" velocity correction *vector* is defined as

$$\Delta\bar{V}_{t_1} = (2\sigma_{\dot{x}_1'})\bar{e}_1 = (62, -27, -18.7) = (\Delta\dot{x}_1, \Delta\dot{y}_1, \Delta\dot{z}_1) \quad (18)$$

where it should be noted that approximately 95% of the velocity corrections called for will have a magnitude less than that given by the formentioned vector. From Appendix B [Eqs. (B8–B10)], the error in the correction can be expressed as

$$\delta\bar{V}_{t_1} = \begin{bmatrix} -\Delta\dot{x}_1\Delta\dot{z}_1\ [(\Delta\dot{x}_1)^2 + (\Delta\dot{y}_1)^2]^{-1/2} \\ -\Delta\dot{y}_1\Delta\dot{z}_1\ [(\Delta\dot{x}_1)^2 + (\Delta\dot{y}_1)^2]^{-1/2} \\ [(\Delta\dot{x}_1)^2 + (\Delta\dot{y}_1)^2]^{-1/2} \end{bmatrix} \delta\theta_{t_1} + \begin{bmatrix} -\Delta\dot{y}_1 \\ \Delta\dot{x}_1 \\ 0 \end{bmatrix} \delta\phi_{t^1} + k_a \begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{y}_1 \\ \Delta\dot{z}_1 \end{bmatrix} + k_b\bar{e}_1 \quad (19)$$

or in numerical form

$$\delta\bar{V}_{t_1} = \begin{bmatrix} 17 \\ -7.5 \\ 67.5 \end{bmatrix} \delta\theta_{t_1} + \begin{bmatrix} 27 \\ 62 \\ 0 \end{bmatrix} \delta\phi_{t_1} + \begin{bmatrix} 62 \\ -27 \\ -18.7 \end{bmatrix} k_a + \begin{bmatrix} 0.883 \\ -0.384 \\ -0.2691 \end{bmatrix} k_b \quad (20)$$

where the "typical" velocity correction vector has been used to estimate $\Delta\bar{V}_{t_1}$. After taking $E(\delta\bar{V}_{t_1}\ \delta\bar{V}_{t_1}{}^T)$ and using the following standard deviations:** $\sigma_{\delta\theta_t} = 0.01$ rad, $\sigma_{\delta\phi_t} = 0.01$ rad, $\sigma_{k_a} = 0.35 \times 10^{-4}$, and $\sigma_{k_b} = 0.01$ fps, the result is

$$E(\delta\bar{V}_{t_1}\delta\bar{V}_{t_1}{}^T) = \begin{bmatrix} (0.319374)^2 & 0.155000 & 0.115000 \\ 0.777140 & (0.624500)^2 & -0.050600 \\ 0.533817 & -0.120119 & (0.674537)^2 \end{bmatrix} = \begin{bmatrix} \sigma_{\dot{x}_1}^2 & \sigma_{\dot{x}_1\dot{y}_1} & \sigma_{\dot{x}_1\dot{y}_1} \\ \rho_{\dot{x}_1\dot{y}_1} & \sigma_{\dot{y}_1}^2 & \sigma_{\dot{y}_1\dot{z}} \\ \rho_{\dot{x}_1\dot{z}_1} & \rho_{\dot{y}_1\dot{z}_1} & \sigma_{\dot{z}_1}^2 \end{bmatrix} \quad (21)$$

The covariance matrix of (21), which gives a reasonable estimate of the accuracy in performing the correction, was added to the precorrection tracking errors as shown in (8) to obtain an a priori covariance matrix of the total error in the state vector. To aid in choosing the time of the second correction, the resultant covariance matrix of miss is estimated. This estimation is computed from

$$B(t_1)E(\delta\bar{V}_{t_1}\ \delta\bar{V}_{t_1}{}^T)B^T(t_1) + \Lambda_{\Delta b_{TR_1}}$$

** These accuracy numbers were used to obtain all the results presented in this paper.

Table 5 Covariance matrix of tracking error 12 hr after injection $(\Lambda_{\Delta b_{TR_1}})^a$

$$\begin{bmatrix} (0.464758E05)^2 & -0.976000E09 & 0 \\ -0.213148 & (0.985241E05)^2 & 0 \\ 0 & 0 & (2)^2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{b_1}^2 & \sigma_{b_1b_2} & \sigma_{b_1V_\infty} \\ \rho_{b_1b_2} & \sigma_{b_2}^2 & \sigma_{b_2V_\infty} \\ \rho_{b_1V_\infty} & \rho_{b_2V_\infty} & \sigma_{V_\infty}^2 \end{bmatrix}$$

[a] All units are in feet, degrees, and feet per second.

Table 6 Matrix relating velocity components of the correction to impact parameter plane coordinates

$$\begin{bmatrix} -0.2173600E06 & -0.1332300E06 & 0.1006000E05 \\ 0.7771000E05 & -0.5163000E05 & 0.1790200E06 \\ 0.1312000E01 & -0.5040000E00 & -0.3450000E00 \end{bmatrix}$$

$$B(t_1) = \begin{bmatrix} \frac{\partial b_1}{\partial\dot{x}_1} & \frac{\partial b_1}{\partial\dot{y}_1} & \frac{\partial b_1}{\partial\dot{z}_1} \\ \frac{\partial b_2}{\partial\dot{x}_1} & \frac{\partial b_2}{\partial\dot{y}_1} & \frac{\partial b_2}{\partial\dot{z}_1} \\ \frac{\partial V_\infty}{\partial x_1} & \frac{\partial V_\infty}{\partial y_1} & \frac{\partial V_\infty}{\partial z_1} \end{bmatrix}$$

Table 7 Covariance matrix of first correction velocity vector

$$\begin{bmatrix} (0.306820E02)^2 & -0.400567E03 & -0.278939E03 \\ -0.958084 & (0.136266E02)^2 & 0.131345E03 \\ -0.920681 & (0.976134 & (0.987452E01)^2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{\dot{x}_1}^2 & \sigma_{\dot{x}_1\dot{y}_1} & \sigma_{\dot{x}_1\dot{z}_1} \\ \rho_{\dot{x}_1\dot{y}_1} & \sigma_{\dot{y}_1}^2 & \sigma_{\dot{y}_1\dot{z}_1} \\ \sigma_{\dot{x}_1\dot{z}_1} & \sigma_{\dot{y}_1\dot{z}_1} & \sigma_{\dot{z}_1}^2 \end{bmatrix}$$

which is the sum of the covariance matrices of miss caused by the correction errors and by the tracking errors. The quantity $B(t_1)E(\delta\bar{V}_{t_1}\ \delta\bar{V}_{t_1}{}^T)B^T(t_1)$ is evaluated using (21) and the matrix of Table 6, and the result is

$$B(t_1)E(\delta\bar{V}_{t_1}\delta\bar{V}_{t_1}{}^T)B^T(t_1) = \begin{bmatrix} 2.04 \times 10^{10} & -0.124 \times 10^{10} & -0.0652 \times 10^5 \\ \text{symmetric} & 1.912 \times 10^{10} & 0.03468 \times 10^5 \\ & & 0.00211 \end{bmatrix} \quad (22)$$

Therefore, using (22) and the matrix of Table 5, the estimated covariance matrix of miss after the correction is

$$\begin{bmatrix} \sigma_{b_1}^2 & \sigma_{b_1b_2} & \sigma_{b_1V_\infty} \\ \rho_{b_1b_2} & \sigma_{b_2}^2 & \sigma_{b_2V_\infty} \\ \rho_{b_1V_\infty} & \rho_{b_2V_\infty} & \sigma_{V_\infty}^2 \end{bmatrix} = \begin{bmatrix} (0.150333E\text{-}06)^2 & -0.222000E\text{-}10 & -0.652000E\text{-}04 \\ -0.870167E\text{-}01 & (0.169706E\text{-}06)^2 & 0.346800E\text{-}04 \\ -0.433704E\text{-}01 & 0.204354E\text{-}01 & (1)^2 \end{bmatrix} \quad (23)$$

The problem now remains to choose the time of the second correction. This time is chosen by a procedure similar to that

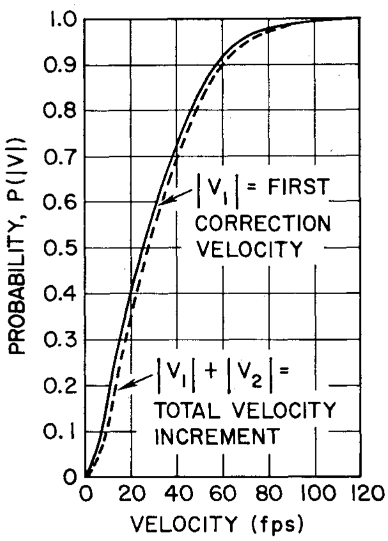

Fig. 2 Cumulative probability of velocity required

used to determine the time of the first correction. The covariance matrix of transformed "burnout" errors is given in (23), and the matrix of (21) when added to the precorrection tracking errors [as in (8)] gives the covariance matrix of the uncertainty of the state vector of the spacecraft. The covariance matrix of tracking errors after 36 hr of tracking from the first midcourse correction (12-hr correction) is

$$\begin{bmatrix} \sigma_{b_1}{}^2 & \sigma_{b_1b_2} & \sigma_{b_1V_\infty} \\ \rho_{b_1b_2} & \sigma_{b_2}{}^2 & \sigma_{b_2V_\infty} \\ \rho_{b_1V_\infty} & \rho_{b_2V_\infty} & \sigma_{V_\infty}{}^2 \end{bmatrix} = \begin{bmatrix} (0.420119E\text{-}05)^2 & -0.171000E\text{-}09 & 0 \\ -0.171542 & (0.237276E\text{-}05)^2 & 0 \\ 0 & 0 & (1)^2 \end{bmatrix} \quad (24)$$

Upon comparing (23) and (24), it is seen that 48 hr from injection is a reasonable time to use as the "fixed time" of the second correction, since the standard deviation of the impact parameter errors caused by performing the first correction is more than three times the corresponding tracking errors. The V_∞ errors are comparable but are so small that they are both negligible.

C. Details of the Monte Carlo Simulation

In this section, the details of the two-correction Monte Carlo simulation are described. A random vector generation program generates samples of the transformed burnout errors, the tracking errors in the two corrections, the orientation errors in the two corrections, the accelerometer error, and the thrust shutdown uncertainty. The computer determines the velocity correction vector to null the indicated error from

$$\bar{V}_{t_1} = -B^{-1}(t_1)\,[\Delta\bar{b}_{BO} + \Delta\bar{b}_{TR}\,(t_1)] = \begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{y}_1 \\ \Delta\dot{z}_1 \end{bmatrix} \quad (25)$$

where $\bar{V}_{t_1} \equiv \Delta\bar{V}_{t_1}$.

The error remaining after the first correction is

$$\delta\bar{\epsilon}_1 = \begin{bmatrix} (\delta b_1)_1 \\ (\delta b_2)_1 \\ (\delta V_\infty)_1 \end{bmatrix} = B(t_1)\left[k_a\bar{V}_{t_1} + k_b\frac{\bar{V}_{t_1}}{|\bar{V}_{t_1}|} + \frac{\partial\bar{V}_{t_1}}{\partial\theta_1}\,\delta\theta_1 + \frac{\partial\bar{V}_{t_1}}{\partial\phi_1}\,\delta\phi_1\right] - \Delta\bar{b}_{TR}(t_1) \quad (26)$$

At the time of the second correction, the indicated error is $\delta\bar{\epsilon}_1 + \Delta\bar{b}_{TR}(t_2)$; hence the second velocity correction vector is computed from

$$\bar{V}_{t_2} = -B^{-1}(t_2)\,[\delta\bar{\epsilon}_1 + \Delta\bar{b}_{TR}(t_2)] = \begin{bmatrix} \Delta\dot{x}_2 \\ \Delta\dot{y}_2 \\ \Delta\dot{z}_2 \end{bmatrix} \quad (27)$$

where $\bar{V}_{t_2} \equiv \Delta\bar{V}_{t_2}$.

As in (26), the error after the second correction is

$$\delta\bar{\epsilon}_2 = \begin{bmatrix} (\delta b_1)_2 \\ (\delta b_2)_2 \\ (\delta V_\infty)_2 \end{bmatrix} = B(t_2)\left[k_a\bar{V}_{t_2} + k_v\frac{\bar{V}_{t_2}}{|\bar{V}_{t_2}|} + \frac{\partial\bar{V}_{t_2}}{\partial\theta_2}\,\delta\theta_2 + \frac{\partial\bar{V}_{t_2}}{\partial\phi_2}\,\delta\phi_2\right] - \Delta\bar{b}_{TR}(t_2) \quad (28)$$

The bracketed expressions in (26) and (28) are identical in form to (19), but in the case of (28) the subscript 2 replaces the subscript 1.

The cumulative probability distribution functions that are presented in Part D of this section were obtained by making 2000 Monte Carlo runs. The computer time necessary to make 2000 runs is now about 1 min.

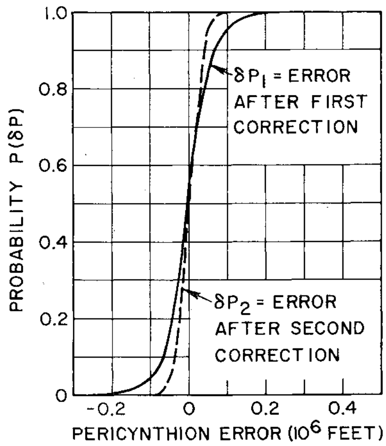

Fig. 4 Cumulative probability of pericynthion error

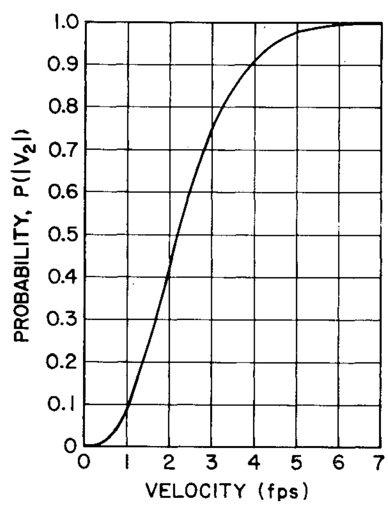

Fig. 3 Cumulative probability of velocity required in second correction

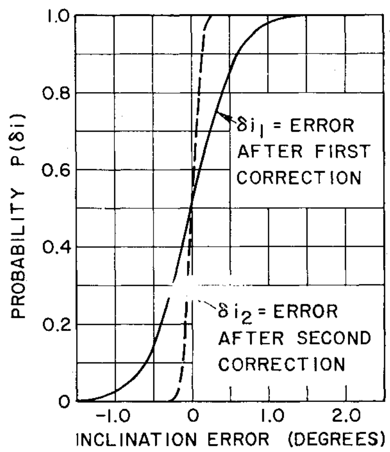

Fig. 5 Cumulative probability of inclination error

Some improvements have been made in the simulation since the results presented in this paper were obtained. These improvements now will be described. Instead of determining $P\{r_i < R_i\}$, where r_i is any one of the random variables of interest, the simulation now determines $P\{|r_i| < |R_i|\}$, since the distribution of the magnitude of the random variables is of more interest than the distribution of the random variable. The discrete point $P\{|r_i| < |R_0|\} = 0.68$ is determined for each of the variables being corrected and the velocity magnitude. This point can be considered as an "equivalent one-sigma" point, since $P\{|r_i| < |1\sigma|\} \cong 0.68$ for a Gaussian random variable. Also computed and printed out is the experimental covariance matrix of

$$[(\delta b_1)_1, (\delta b_2)_1, (\delta V_\infty)_1], [(\delta b_1)_2, (\delta b_2)_2, (\delta V_\infty)_2], |\bar{V}_{t_1}|, |\bar{V}_{t_2}|, |\bar{V}_{t_1}| + |\bar{V}_{t_2}|, \delta\bar{V}_{t_2}$$

Of course this covariance matrix does not define the joint probability density function of the random variables since they are non-Gaussian. However, this matrix is useful when it is expedient to make a Gaussian approximation. For example, a problem of interest is to determine how accurately the trajectory can be re-established after the midcourse guidance phase. To solve this problem, an a priori covariance matrix of the uncertainty in the state vector of the spacecraft after the last correction is necessary. The matrix sum of the covariance matrix of tracking errors before the last correction and the experimental covariance matrix of $\delta\bar{V}_{t_2}$ furnishes an excellent approximation to the desired a priori matrix. The Gaussian probability density function defined by the approximate a priori matrix should be a good approximation to the actual density function, since the tracking errors are Gaussian (at least, they are assumed to be Gaussian) and the components of $\delta\bar{V}_{t_2}$ have no odd-order statistical moments.

It is important to note that this simulation is based on linear theory. Some preliminary work has been performed concerning the analysis of the midcourse guidance phase of missions in which severe nonlinearities are present and is presented in Sec. III.

D. Results of the Monte Carlo Simulation

The data necessary to simulate the midcourse guidance phase of this mission already have been presented. The cumulative probability distributions that were obtained are shown in Figs. 2–6 and are based on 2000 runs. Since the simulation used impact-parameter plane variables, it was desirable to convert these variables to pericythion error and inclination error. Approximate conversion factors are

$$\delta P \approx 0.70\ \delta b_1 = \text{pericythion error}$$

$$\delta i = [57.3/(15 \times 10^6)]\delta b_2 = \text{inclination error}$$

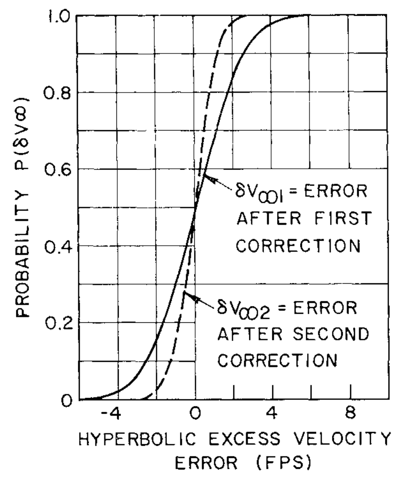

Fig. 6 Cumulative probability of the hyperbolic excess velocity error

Table 8 "Equivalent" one-sigma accuracies and midcourse velocity requirements for an earth-to-moon flight

V_1	= 38.5 fps
V_2	= 2.7 fps
$V_1 + V_2$	= 39.5 fps
δP_1	= 50,000 ft
δi_1	= 0.5 deg
$\delta V_{\infty 1}$	= 2.3 fps
δP_2	= 34,000 ft
δ_{i2}	= 0.11 deg
$\delta V_{\infty 2}$	= 1 fps

The "equivalent" 1σ accuracies and fuel requirements are shown in Table 8.

III. Simulation of Missions in Which Nonlinearities Are Present

In this section, a proposed Monte Carlo simulation is described which can be used to determine the fuel requirements and final accuracy of the midcourse guidance phase of missions in which nonlinearities are present. Such nonlinearities may arise from several possible sources. For initial missions of a spacecraft returning from the moon, the major contribution to the nonlinearity will be due to the guidance system. It is logical to assume that, for initial missions, sophistication of the guidance system will be restricted due to weight limitations. Thus, the larger expected burnout errors will involve second-order variations of the terminal parameters. Additional contributions to this error will be made by the uncertainty in the mass and shape of the moon.

Concerning the terminal parameters, it is reasonable to expect that the latitude and longitude at the re-entry point and the re-entry angle will be important. Satisfaction of certain values of these conditions will be necessary if the vehicle is to survive and land at a designated point on the earth. These parameters, however, in part because of the sphericity of the earth, do *not* vary linearly with respect to the lunar burnout and midcourse variables for the guidance system expected to be used in initial missions. Furthermore, no suitable representation of these terminal parameters has been found, such as the impact parameter plane discussed and used in Sec. II, which will linearize the problem.

The following discussion illustrates a scheme that may be used for a Monte Carlo simulation of a one-correction moon-to-earth flight when either two or three terminal variables are to be controlled.

A. Basic System for the Fuel and Error Analysis

The basic system that will be used to simulate the mission is shown in Fig. 7. A Gaussian-random-vector generation program produces a sample of the burnout errors, the tracking errors, and the midcourse correction errors when the covariance matrix of these errors is specified. The trajectory program computes the actual conditions that would occur at midcourse and then adds the tracking errors to compute the apparent conditions at midcourse. The optimum velocity correction scheme computes the direction and magnitude of the minimum velocity correction on the basis of the apparent conditions. Since the errors in the midcourse correction generally depend on the direction and magnitude of the velocity correction, $\delta(\Delta\bar{V})$ cannot be computed until $\Delta\bar{V}$ has been decided upon. After $\delta(\Delta\bar{V})$ has been computed, the actual conditions before the midcourse maneuver, $\Delta\bar{V}$ and $\delta(\Delta\bar{V})$, are added to obtain the resultant position and velocity coordinates. The trajectory program then computes values for the controlled end conditions and compares them with the desired end conditions and, on the basis of the specified tolerances, decides whether or not a success or failure

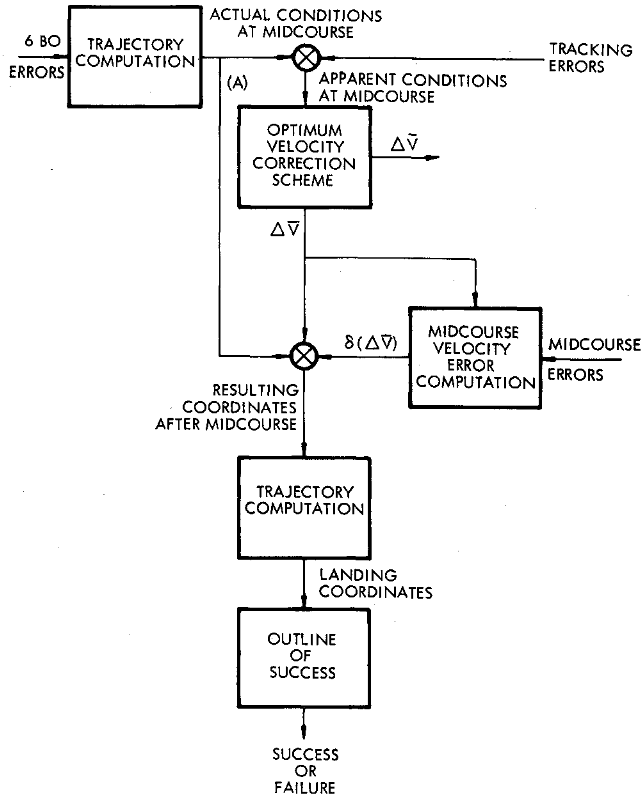

Fig. 7 Basic system for the simulation of the moon-to-earth mission

has occurred.†† The amount of fuel used in midcourse is stored in the machine for the future computation of points on the cumulative distribution function of required fuel.

B. Technique Used for the Trajectory Computation

If the Monte Carlo technique discussed here is to produce results that have a high degree of confidence, many trajectory simulations are required. This presupposes a very-high-speed trajectory calculation procedure that is available only if it is based on an analytic or two-body model. One such model concerning lunar trajectories which is adaptable to this problem first was discussed in a paper by Egorov[2] in 1956. For this model, Egorov assumes that the moon is enclosed in a sphere called the "sphere of action" within which only the influence of the moon is considered. Outside of this sphere, the earth is considered to be the sole attracting body. The radius of this sphere, which is calculated to be about 30,000 naut miles, was assumed by Egorov to be that distance at which the perturbing force due to the earth when the moon is considered the central body is equal to the perturbing force due to the moon when the earth is considered the central body.

Space Technology Laboratories has extended Egorov's assumptions to include three-dimensional lunar trajectories and has written a program that will search for moon-to-earth trajectories satisfying certain end conditions.[7] These conditions are the total time of flight, the impact latitude, longitude, and flight-path angle, and the selenographic latitude and longitude of the launch site. The program solves for the trajectory by an iteration process that attempts to match the moon phase and earth phase conics at the interface or sphere of action. The resulting launch time, burnout position, and velocity then can be supplied directly into the midcourse guidance simulation program, which must, of necessity, contain the same two-body equations. This describes the analytic model upon which the midcourse analysis will be based and a method by which nominal trajectories may be generated.

Once a nominal trajectory has been found, the midcourse guidance simulation program will add the burnout errors to the nominal initial conditions and then calculate the midcourse position and velocity coordinates at the desired point along the trajectory. It can do this in the following manner:

1) The six selenographic position and velocity coordinates are transformed to the selenographic Cartesian system. Polar coordinates are inputs to the program, since this is the form in which the burnout errors will be introduced.

2) The selenographic Cartesian coordinates then are transformed to the equatorial system by means of three Euler angles that describe the librations of the moon. The instantaneous transformation used is determined by the time of launch, which is obtained from the forementioned search program.

3) The equatorial position and velocity vectors then are used to determine the conic elements of the moon phase trajectory. Also, the transformation from in-plane to the equatorial system is found.

4) With this transformation, it is possible to determine the trajectory position and velocity vectors at the moon-earth transfer point. Calculating the time that the vehicle passes through this point, it is possible to obtain the position and velocity of the moon at this time and hence to determine the position and velocity of the vehicle with respect to the earth.

5) These will be the initial conditions of the earth phase conic from which, as in point 3, the conic elements and the in-plane to equatorial transformation may be found.

6) Specifying the distance from the earth at which the velocity correction is to be made, the midcourse conditions may be calculated.

At this point the *actual* midcourse conditions have been found. To these position and velocity vectors are added the tracking errors that are obtainable from the Gaussian random vector generation program. The resulting vehicle position and velocity represent the *apparent* midcourse conditions that are used to compute the optimum midcourse correction velocity. This correction velocity and the midcourse errors then are added to the actual initial midcourse velocity, and the resulting velocity together with the actual position are used to calculate the vehicle terminal conditions.

The calculation of the trajectory from midcourse to termination will involve essentially the same procedure discussed in paragraphs 5 and 6. That is, the actual position and velocity vectors are used to calculate the midcourse to termination conic elements and the in-plane to equatorial transformation. These, in turn, may be used to find perigee, impact latitude and longitude, re-entry flight path angle, or whatever the desired set of terminal parameters happen to be for the problem considered. Success or failure in meeting prespecified values of these parameters then can be determined.

C. Computation of the Velocity Correction

1. *Latitude and longitude control*

Before going into the details of the scheme proposed in this paper for determining the optimum velocity correction, a short digression into the geometrical aspect of the problem will be presented. The perturbations in latitude and longitude of the spacecraft at impact caused by the midcourse correction can be expressed symbolically as

$$\Delta_{\text{lat}} = f(\Delta\dot{x},\Delta\dot{y},\Delta\dot{z}) \tag{29}$$

$$\Delta_{\text{long}} = g(\Delta\dot{x},\Delta\dot{y},\Delta\dot{z}) \tag{30}$$

where $(\Delta\dot{x},\Delta\dot{y},\Delta\dot{z})$ is the midcourse correction vector represented in earth-centered, inertial coordinates.

†† Cumulative probability functions also could be determined.

Normally the surfaces in $(\Delta\dot{x},\Delta\dot{y},\Delta\dot{z})$ space represented by (29) and (30) intersect in a curve. The coordinates of *any* point on the curve represent the components of a correction vector that will null the indicated errors in latitude and longitude. The problem is to find the coordinates of the point on the curve which is closest to the origin. The special case that is obtained when f and g are linear functions is derived in Appendix C.

When the nonlinear, moon-to-earth midcourse correction problem is considered, explicit expressions relating Δ_{lat} and Δ_{long} to the components of the correction cannot be determined. Since explicit expressions cannot be found, the approach used will be to determine many of the possible solutions and to use a convergence routine to find the optimum solution. (Linear theory could be used for a first trial.)

The method of computing the optimum velocity correction now will be explained. The apparent position and velocity of the spacecraft at midcourse as seen by an earth-based observer are computed from the known nominal trajectory and the error sample that was generated by the random vector generation program. A time of flight (t_f) is chosen, and the computer determines the position vector of the target t_f hours after midcourse. The time of flight is related to the semimajor axis by Lambert's theorem.[3] That is,

$$t_f = (a^{3/2}/\mu^{1/2})[N - \sin N - (N_1 - \sin N_1)] \qquad (31)$$

where

$$\sin\frac{1}{2}N = \frac{1}{2}\left[\frac{|\bar{r}_M| + |\bar{r}_f| + |\bar{r}_M - \bar{r}_f|}{a}\right]^{1/2} \qquad (32)$$

and

$$\sin\frac{1}{2}N_1 = \frac{1}{2}\left[\frac{|\bar{r}_M| + |\bar{r}_f| - |\bar{r}_M - \bar{r}_f|}{a}\right]^{1/2} \qquad (33)$$

Since

$$|\bar{r}_M| + |\bar{r}_f| \approx |\bar{r}_M - \bar{r}_f| \qquad (34)$$

then

$$N_1 - \sin N_1 \approx 0 \qquad (35)$$

and t_f can be expressed as

$$t_f \approx (a^{3/2}/\mu^{1/2})[N - \sin N] = F(|\bar{r}_M| + |\bar{r}_f| + |\bar{r}_M - \bar{r}_f|, a) \qquad (36)$$

The function F of (36) could be stored in the computer in the form of tables. An interpolation routine then could be used to obtain the semimajor axis after t_f is specified and $|\bar{r}_M| + |\bar{r}_f| + |\bar{r}_M - \bar{r}_f|$ is computed.

After the semimajor axis has been computed, the desired velocity is determined from

$$V_M = [(2\mu/r_M) - (\mu/a)]^{1/2} \qquad (37)$$

and then the desired flight path angle is specified by

$$\beta_M = \frac{1}{2}\tan^{-1}\left[\frac{(r_M/r_f) - \cos\psi_f}{\sin\psi_f}\right] + \frac{1}{2}\sin^{-1}\left[\frac{(\psi_f/\lambda_M)\sin^2(\psi_f/2) - [(r_M/r_f) - \cos\psi_f]}{\{\sin^2\psi_f + [(r_M/r_f) - \cos\psi_f]^2\}^{1/2}}\right] \qquad (38)‡‡$$

where

$$\lambda_M = r_M V_M{}^2/\mu \qquad (39)$$

and

$$\psi_f = \cos^{-1}\left(\frac{\bar{r}_M \cdot \bar{r}_f}{|\bar{r}_M||\bar{r}_f|}\right) \qquad (40)$$

‡‡ The machine will have to determine the proper quadrant of β_M.

Equation (37) relates the kinetic and potential energy of the orbit, whereas (38) is derived by solving[4]

$$\frac{r_M}{r_f} = \frac{1 - \cos\psi_f}{\lambda_M \sin^2\beta_M} + \frac{\sin(\beta_M - \psi_f)}{\sin\beta_M} \qquad (41)$$

for β_M.

Since the velocity vector must be in the plane of $\bar{r}_f$ and $\bar{r}_M$, (37) and (38) completely specify a velocity vector that will insure that the spacecraft re-enters at the desired latitude and longitude. This velocity is given by

$$\bar{V}_M = V_M\left[\frac{\sin\beta_M}{\sin\psi_f}\left(\frac{\bar{r}_f}{r_f}\right) - \left(\cos\beta_M - \frac{\sin\beta_M}{\tan\psi_f}\right)\left(\frac{\bar{r}_M}{r_M}\right)\right] \qquad (42)$$

By subtracting the initial midcourse velocity vector from $\bar{V}_M$, the correction velocity vector $\Delta\bar{V}$ can be determined. By using the procedure just described, a curve of $|\Delta\bar{V}|$ vs t_f can be computed, and the minimum velocity increment can be determined from that curve. (Actually, a convergence routine would be used to find the optimum solution.) Therefore the output of this stage of the simulation will be the desired direction of the spacecraft body axis and the magnitude of the velocity correction.

If hyperbolic approach trajectories are used, the only modifications to the foregoing discussion occur in (31–37). The modified equations are obtained by replacing Eqs. (36) and (37) by

$$t_f = (a^{3/2}/\mu^{1/2})[\sinh N - N] \qquad (36a)$$

and

$$v_0 = [(2\mu/r_M) + (\mu/a)]^{1/2} \qquad (37a)$$

2. *Latitude, longitude, and re-entry flight-path angle control*

Since three re-entry variables are being controlled and there are only three degrees of freedom in the performance of the midcourse correction (it is assumed that the correction is applied at a fixed distance from the earth), there is assumed§§ to be only one possible velocity correction vector. The problem considered here is to determine that vector.

The method of computing the direction and magnitude of the midcourse velocity increment now will be explained. A trial value for the time of flight is selected, and the position vector of the target t_f hours after midcourse is determined. Equation (36) is used to compute the semimajor axis of the trial orbit, the velocity is found from (37) and β_M is determined by (38–40). From conservation of angular momentum and conservation of energy, the re-entry flight-path angle can be computed. If the computed flight-path angle is not equal to the desired value, another time of flight is chosen. An iteration technique would be used to converge on the correct time of flight. Therefore, the output of this option of the simulation is the correction vector $\Delta\bar{V}$.

D. Estimation of Computing Time to Perform Simulation

In estimating the running time of the simulation, it is possible to break down the machine calculations into the following three categories: 1) trajectory determination, 2) optimum velocity correction determination, and 3) Gaussian-random-vector generation.

These represent the major computation groups. The time required for other machine operations and calculations, such as reading the ephemeris tape (once per set of runs), computation of midcourse performance errors, and evaluation of fuel requirements and mission success, will be minor com-

§§ It is possible that there will be more than one solution. In this case, the computer will choose the solution that has its time of flight closest to nominal.

pared to these. The machine time estimate for the first and second groups is based on an actual count of operations required in the computation.

The first, or trajectory determination group, represents the calculations required to go from lunar burnout to midcourse plus those required after the midcourse correction to impact. These calculations consist of about 75 major operations, i.e., those requiring subroutines such as sines, square roots, etc., and 400 minor operations, such as multiplication and addition. In this case, the minor operations will consume only about 5% of the time that the major operations will take. The 7090 machine time required to calculate this group amounts to about 0.20 sec/trajectory.

The operation time for the second group or the optimum velocity correction determination also can be determined by an operation count. This estimation, however, will be less precise because the calculation requires two iteration processes. First, an iteration is required to solve Lambert's implicit expression. This will require 0.0065 sec/loop, or, if approximately 12 loops are required per solution, then the time will be about 0.08 sec. Once Lambert's expression has been solved, about an additional 0.02 sec will be necessary to find a midcourse correction velocity computation. Second, since the velocity correction just calculated is very likely not *optimum*, additional velocity correction calculations will have to be made and the optimum chosen from this set. If it is estimated that a set of eight velocity calculations is required, then a total of 0.80 sec will be consumed by this group.

The method by which the computing time for the third group has been found is essentially empirical, that is, by actually running the random vector generation program on the 7090 and estimating the running time. For latitude and longitude control and a single midcourse correction, it will be necessary to generate 16 random variables. The estimate of the running time in this case amounts to 0.12 sec/run.

Estimation of the total computing time based on the foregoing amounts to about 1.2 sec/run, where 0.08 sec has been allowed for minor machine operations. It is clear from the foregoing that the majority of the machine time is consumed in the calculation of the optimum velocity correction, i.e., about two thirds of the total time. There are three possible alternative methods that may be used to reduce this running time. First, it may be possible to find an analytic solution or approximation to the optimum midcourse velocity requirement. A second method would be to find rapidly converging series for some of the calculations performed now. This would replace the relatively slow routines that presently must be used. A final possibility would be to solve Lambert's expression by means of a precalculated table. This table would be a function of two independent variables and would require a double interpolation scheme. It is felt that this would be the best approach to decreasing the simulation running time.

IV. Discussion of Statistical Aspects of Monte Carlo Technique‖ ‖

In analyzing the data obtained from the simulation, the computer can determine cummulative probability curves (or surfaces) or calculate the experimental probability of success. When a probability of success computation is desired, the machine decides, on the basis of a specified criterion, whether or not a success has occurred by inspecting the final values of the controlled variables. After completing a required number of runs, the machine counts the number of successes obtained and divides by the number of runs. The number obtained is the estimated probability of success. How meaningful the estimation is will be discussed now.

‖ ‖ For a more thorough discussion of the statistics of the Monte Carlo Technique, see Ref. 5.

Since the simulation is estimating a binomial event (success or failure), $\hat{P}_0$ (estimated probability of success) is asymptotically Gaussian, probability statements about $\hat{P}_0$ are relatively easy to make. A meaningful statement to make about $\hat{P}_0$ is

$$P\{|\hat{P}_0 - P_0| < d\} = \alpha \quad (43)$$

(In words, there is a $100\alpha\%$ probability that the absolute value of the difference between the true and estimated probability of success will be less than d.) From Ref. 5, the number of runs necessary to obtain this degree of confidence is# #

$$n = P_0(1 - P_0)(Z\alpha^2/d^2) \quad (44)$$

where

$$\alpha = \frac{1}{(2\pi)^{1/2}} \int_{-Z_\alpha}^{Z_\alpha} e^{-(1/2)t^2}\, dt \quad (45)$$

Since $P_0(1 - P_0) \leq \frac{1}{4}$, (44) can be expressed as

$$n \leq Z_\alpha{}^2/4d^2 \quad (46)$$

Therefore, by making 1000 runs, the probability of success can be estimated within $\pm 3\%$ with more than 95% confidence in the result. (Note that the substitution $P_0(1 - P_0) = \frac{1}{4}$ is very pessimistic when $P_0 \approx 0.9$.)

When a complete cumulative distribution function (c.d.f.) is desired, the number of computer runs necessary to get a desired degree of confidence becomes very large. For example, suppose it is required to estimate the c.d.f. of the midcourse velocity increment. To estimate this c.d.f. within $\pm 3\%$ ($d = 0.03$) with a 95% confidence would take about 2500 runs.[8] If only a point on the curve which corresponds to a probability of approximately 0.9 is being estimated, about 400 runs are required to obtain the same degree of confidence.

An important characteristic of the Monte Carlo method as applied to this type of simulation is that the number of runs for a given level of confidence is independent of the complexity of the simulation (i.e., the number of midcourse corrections, number of error sources considered, etc.). The penalty paid for complexity is the machine time per run.

Appendix A: Probability-of-Success Integral for an Attitude-Controlled Spacecraft

Let $\Delta\bar{X}$ represent the errors in the controlled terminal variables, and let B be a 3×3 matrix that transforms the midcourse velocity correction vector into changes in $\bar{X}$. Therefore $\Delta\bar{X} = B\Delta\bar{V}$, and

$$\delta(\Delta\bar{X}) = B(\partial\Delta\bar{V}/\partial\theta)\delta\theta + B(\partial\Delta\bar{V}/\partial\phi)\delta\phi + B\delta(\Delta\bar{V}) + \delta\bar{X}_{\text{track}}$$

where $\delta\theta$ and $\delta\phi$ are the body axis orientation errors.*** It is assumed that $\delta(\Delta\bar{V})$ can be expressed

$$\delta(\Delta\bar{V}) = k_a\Delta\bar{V} + k_b(\Delta\bar{V}/|\Delta\bar{V}|)$$

and hence

$$\delta(\Delta\bar{X}) = B(\partial\Delta\bar{V}/\partial\theta)\delta\theta + B(\partial\Delta\bar{V}/\partial\phi_1)\delta\phi + [k_aB + (k_b/\Delta V)]\Delta\bar{V} + \delta\bar{X}_{\text{track}}$$

where $\delta\theta$, $\delta\phi$, k_a, and k_b are independent Gaussian random variables.

Equations (44) and (45) are based on the fact that P_0 is approximately Gaussian with mean = P_0 and variance = $[P_0(1 - P_0)]/n$.

*** See Fig. 8 and Eqs. (B8–B10) for $(\partial\Delta\bar{V}/\partial\theta)\delta\theta + (\partial\Delta\bar{V}/\partial\phi)\delta\phi$.

If S is the volume of success in $\Delta\bar{X}$ space, the probability of success is

$$P(S) = \int_{\text{all } \Delta V} \left[\int_S p_{\Delta X}\,[\delta(\Delta\bar{X})|\Delta\bar{V}]dx_1\,dx_2\,dx_3\right] \times p_{\Delta V}\,(\Delta\bar{V})dV_1\,dV_2\,dV_3$$

$$= \int_{\text{all } \Delta X} P(S|\Delta\bar{V})\; p_{\Delta V}\,(\Delta\bar{V})dV_1\,dV_2\,dV_3$$

Since $p_{\Delta X}[\delta(\Delta\bar{X})|\Delta\bar{V}]$ and $p_{\Delta V}(\Delta\bar{V})$ are both trivariate Gaussian density functions, $P(S)$ involves a six-dimensional integration.

Appendix B: Computation of Errors in the Performance of the Midcourse Correction

After the midcourse correction vector has been determined, the errors that result from the imperfect performance of the correction can be computed. The errors in the correction are usually of two types: angular position errors caused by reorienting the body axis and velocity magnitude errors. The angular position errors are dependent on the amount of maneuvering, and the velocity error is a function of the magnitude of the correction.

An error analysis of the midcourse correction now will be performed. From Fig. 8, the components of the velocity correction are

$$\Delta\dot{x} = \Delta V\cos\theta\cos\phi \quad \text{(B1)}$$

$$\Delta\dot{y} = \Delta V\cos\theta\sin\phi \quad \text{(B2)}$$

$$\Delta\dot{z} = \Delta V\sin\theta \quad \text{(B3)}$$

After taking variation in (B1–B3),

$$\delta(\Delta\dot{x}) = \Delta\dot{x}[\delta(\Delta V)/\Delta V] - \Delta\dot{z}\cos\phi\;\delta\theta - \Delta\dot{y}\;\delta\phi \quad \text{(B4)}$$

$$\delta(\Delta\dot{y}) = \Delta\dot{y}[\delta(\Delta V)/\Delta V] - \Delta\dot{z}\sin\phi\;\delta\theta + \Delta\dot{x}\;\delta\phi \quad \text{(B5)}$$

$$\delta(\Delta\dot{z}) = \Delta\dot{z}[\delta(\Delta V)/\Delta V] + \Delta V\cos\theta\;\delta\theta \quad \text{(B6)}$$

In the simulation, it will be assumed that $\delta(|\Delta\bar{V}|)$ can be expressed as

$$\delta(|\Delta\bar{V}|) = k_a|\Delta\bar{V}| + k_b \quad \text{(B7)}$$

and that the $\delta\theta$ and $\delta\phi$ are independent of the maneuvering. The trigonometric functions of (B4–B6) can be expressed in terms of the velocity components, and the result is

$$\delta(\Delta\dot{x}) = \Delta\dot{x}\left[k_a + \frac{k_b}{\Delta V}\right] - \Delta\dot{z}\,\frac{(\Delta\dot{x})\;\delta\theta}{[(\Delta\dot{x})^2 + (\Delta\dot{y})^2]^{1/2}} - \Delta\dot{y}\;\delta\phi \quad \text{(B8)}$$

$$\delta(\Delta\dot{y}) = \Delta\dot{y}\left[k_a + \frac{k_b}{|\Delta\bar{V}|}\right] - \Delta V_z\,\frac{(\Delta\dot{y})\;\delta\theta}{[(\Delta\dot{x})^2 + (\Delta\dot{y})^2]^{1/2}} + (\Delta\dot{x})\;\delta\phi \quad \text{(B9)}$$

$$\delta(\Delta\dot{z}) = \Delta\dot{z}\left[k_a + \frac{k_b}{|\Delta\bar{V}|}\right] + [(\Delta\dot{x})^2 + (\Delta\dot{y})^2]^{1/2}\,\delta\theta \quad \text{(B10)}$$

It is assumed that $\delta\theta$, $\delta\phi$, k_a, and k_b are independent Gaussian random variables.

For the preliminary analysis that this type of simulation would be required to perform, it is felt that (B4–B6) are reasonably accurate representations of the true situation. They allow the analyst to consider the effects of the final orientation accuracy as well as the effects of velocity errors.

Appendix C: Simulation of Missions Where Two Terminal Variables Are Controlled

It is the purpose of this appendix to show how the equations presented in Part C of Sec. II should be modified when only two terminal variables are controlled. If only two terminal variables are controlled, an optimization problem is present since three degrees of freedom, the three components of velocity, are available. Let b_1 and b_2 be the indicated error in the two terminal variables under control, and then the equations relating these variables to the correction velocity are

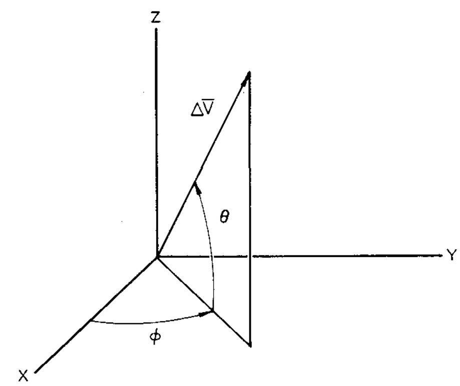

Fig. 8 Orientation of the body axis for the midcourse correction

$$-b_1 = (\partial b_1/\partial\dot{x})\Delta\dot{x} + (\partial b_1/\partial\dot{y})\Delta\dot{y} + (\partial b_1/\partial\dot{z})\Delta\dot{z} \quad \text{(C1)}$$

$$-b_2 = (\partial b_2/\partial\dot{x})\Delta\dot{x} + (\partial b_2/\partial\dot{y})\Delta\dot{y} + (\partial b_2/\partial\dot{z})\Delta\dot{z} \quad \text{(C2)}$$

or in matrix form

$$-\bar{b} = -\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial b_1}{\partial\dot{x}} & \frac{\partial b_1}{\partial\dot{y}} & \frac{\partial b_1}{\partial\dot{z}} \\ \frac{\partial b_2}{\partial\dot{x}} & \frac{\partial b_2}{\partial\dot{y}} & \frac{\partial b_2}{\partial\dot{z}} \end{bmatrix} \begin{bmatrix} \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{z} \end{bmatrix} = \frac{\partial\bar{b}}{\partial\bar{V}}\,\Delta\bar{V} \quad \text{(C3)}$$

The optimum correction velocity vector is considered to be that correction vector which satisfies the constraints (C1) and (C2) and has the minimum $\Delta\bar{V}^2 = (\Delta\dot{x})^2 + (\Delta\dot{y})^2 + (\Delta\dot{z})^2$. After using the standard Lagrange parameter approach, the result is

$$2\begin{bmatrix} \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{z} \end{bmatrix} = \begin{bmatrix} \frac{\partial b_1}{\partial\dot{x}} & \frac{\partial b_2}{\partial\dot{x}} \\ \frac{\partial b_1}{\partial\dot{y}} & \frac{\partial b_2}{\partial\dot{y}} \\ \frac{\partial b_1}{\partial\dot{z}} & \frac{\partial b_2}{\partial\dot{z}} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad \text{(C4)}$$

or

$$2\Delta\bar{V} = (\partial\bar{b}/\partial\bar{V})^T\bar{\lambda} \quad \text{(C5)}$$

where λ_1 and λ_2 are the Lagrange parameters. Multiplying (C5) by $(\partial\bar{b}/\partial\bar{V})$ gives

$$2(\partial\bar{b}/\partial\bar{V})\Delta\bar{V} = -2\bar{b} = (\partial\bar{b}/\partial\bar{V})(\partial\bar{b}/\partial\bar{V})^T\bar{\lambda}$$

or

$$\bar{\lambda} = -2[(\partial\bar{b}/\partial\bar{V})(\partial\bar{b}/\partial\bar{V})^T]^{-1}\,\bar{b}$$

$$\Delta\bar{V} = -(\partial\bar{b}/\partial\bar{V})^T[(\partial\bar{b}/\partial\bar{V})(\partial\bar{b}/\partial\bar{V})^T]^{-1}\,\bar{b} \quad \text{(C6)}$$

The equations that are used in the simulation when only two terminal variables are controlled are

$$\bar{V}_{t_1} = -B^T(t_1)[B(t_1)B^T(t_1)]^{-1}[\Delta\bar{b}_{BO} + \Delta\bar{b}_{TR}(t_1)] \quad \text{(C7)}$$

$$\delta\bar{\epsilon}_1 = B(t_1)[k_a\,\bar{V}_{t_1} + k_b(\bar{V}_{t_1}/|\bar{V}_{t_1}|) + (\partial\bar{V}_{t_1}/\partial\theta_1)\delta\theta_1 + (\partial\bar{V}_{t_1}/\partial\phi_1)\delta\phi_1] - \Delta\bar{b}_{TR}(t_1) \quad \text{(C8)}$$

$$\bar{V}_{t_2} = -B^T(t_2)\,[B(t_2)B^T(t_2)]^{-1}[\delta\bar{\epsilon}_1 + \Delta\bar{b}_{TR}(t_2)] \quad \text{(C9)}$$

$$\delta\bar{\epsilon}_2 = B(t_2)[k_a\bar{V}_{t_2} + k_b(\bar{V}_{t_2}/|\bar{V}_{t_2}|) + (\partial\bar{V}_{t_2}/\partial\theta_2)\delta\theta_2 + (\partial\bar{V}_{t_2}/\partial\phi_2)] - \Delta\bar{b}_{TR}(t_2) \quad \text{(C10)}$$

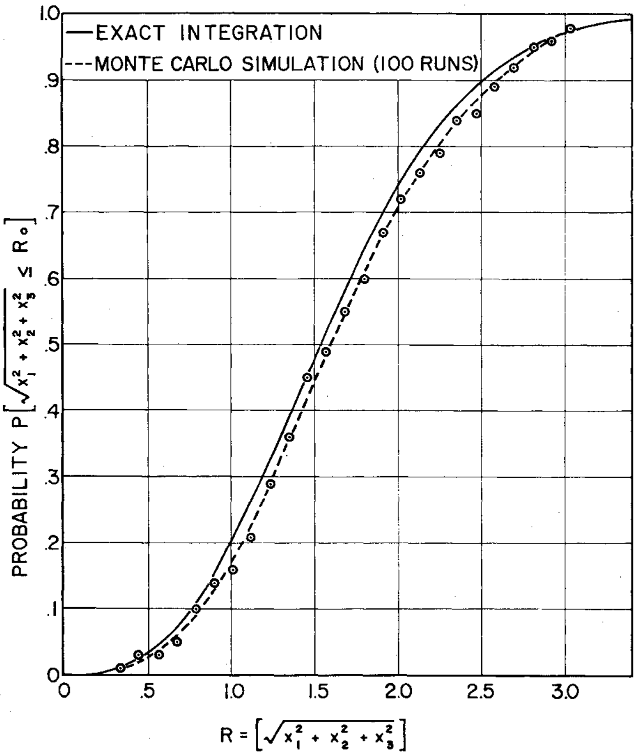

Fig. 9 Cumulative probability of $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

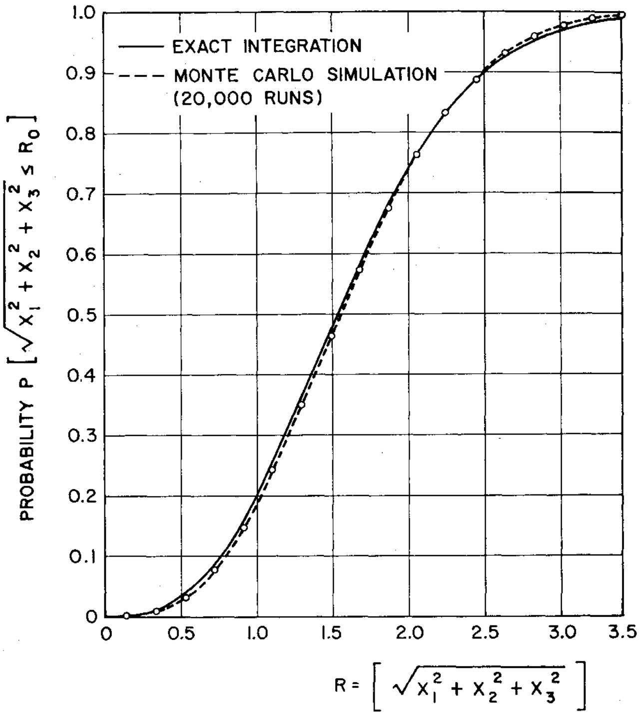

Fig. 11 Cumulative probability of $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

Appendix D: Comparison of Monte Carlo and Direct-Integration Techniques

In this appendix, two relatively simple statistical problems are solved by both the direct-integration approach and the Monte Carlo technique. The first of these problems involves determining

$$P[(x_1^2 + x_2^2 + x_3^2)^{1/2} \leq R_0] \tag{D1}$$

where the x's are standard, uncorrelated Gaussian variables.

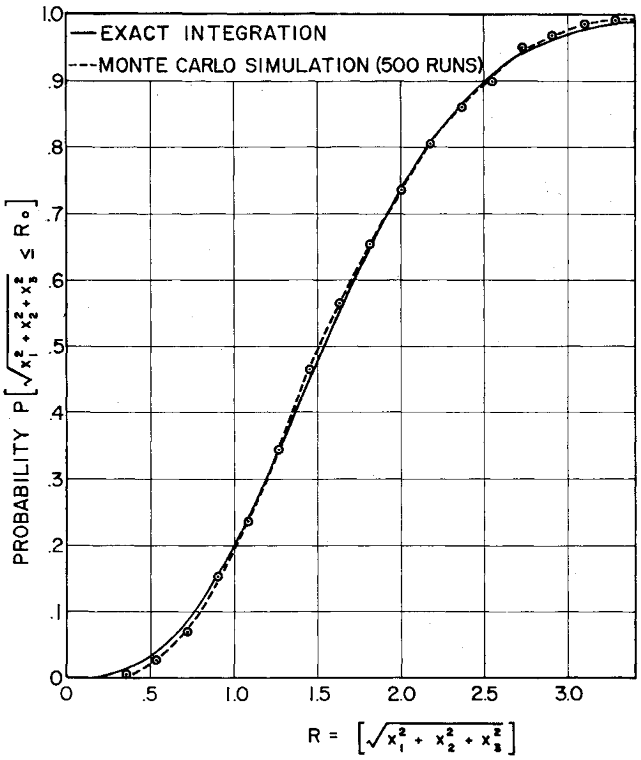

Fig. 10 Cumulative probability of $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

The solution of (D1) can be expressed in terms of tabulated functions as follows:

$$P = \frac{1}{(2\pi)^{3/2}} \iiint_{(x_1^2+x_2^2+x_3^2)^{1/2} \leq R_0} e^{-1/2(x_1^2+x_2^2+x_3^2)}\, dx_1\, dx_2\, dx_3$$

$$= \frac{1}{(2\pi)^{1/2}} \int_0^{R_0} \int_0^{\pi} r^2 \sin\theta e^{-r^2/0}\, d\theta dr$$

$$= 2\left\{\int_0^{R_0} \frac{e^{-r^2/2}}{(2\pi)^{1/2}}\, dr - R_0 \left[\frac{e^{-R_0^2/2}}{(2\pi)^{1/2}}\right]\right\} \tag{D2}$$

A plot of (D2) as a function of R_0 is shown in Figs. 9–11. Also plotted on the graphs are the results of a Monte Carlo simulation.

A problem of interest in the statistical analyses in lunar satellite trajectories is that of determining

$$P[-|\delta P_{e_0}| \leq \delta P_e \leq \delta A_p \leq |\delta A_{P_0}|] \tag{D3}$$

where δP_e and δA_p are the errors in pericynthion and apocynthion. It is shown in Ref. 6 that for a nominal circular orbit

$$\delta P_e = x_1 - (x_2^2 + x_3^2)^{1/2} \tag{D4}$$

$$\delta A_p = x_1 + (x_2^2 + x_3^2)^{1/2} \tag{D5}$$

where

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 2(r_0/v_0) & 0 \\ 1 & 2(r_0/v_0) & 0 \\ 0 & 0 & r_0 \end{bmatrix} \begin{bmatrix} \delta r_0 \\ \delta V_0 \\ \delta\beta_0 \end{bmatrix} \tag{D6}$$

The variables r_0, V_0, and β are the nominal selenocentric radius, velocity, and flight-path angle. A numerical solution of (D3) was obtained by integrating the trivariate Gaussian density of (x_1, x_2, x_3) over the conical volume described by (D4) and (D5). The assumed covariance matrix of (x_1, x_2, x_3) is shown in Table 9 for a 600,000-ft-altitude, circular, lunar parking orbit.

A plot of the results of the numerical integration is shown in Fig. 12. Also shown is the result of 28,000 Monte Carlo runs. It can be seen that good agreement is obtained. The

Table 9 Covariance matix of (x_1,x_2,x_3)

$$\begin{bmatrix} (0.184202E05)^2 & 0.305534E\text{-}09 & 0.123989E\text{-}10 \\ 0.999830 & (0.165897E\text{-}05)^2 & 0.111474E\text{-}10 \\ 0.984508 & 0.982799 & (0.683708E05)^2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{x_1}{}^2 & \sigma_{x_1x_2} & \sigma_{x_1x_3} \\ \rho_{x_1x_2} & \sigma_{x_2}{}^2 & \sigma_{x_2x_3} \\ \rho_{x_1x_3} & \rho_{x_2x_3} & \sigma_{x_3}{}^2 \end{bmatrix}$$

numerical integration took 3.5 min of machine time, whereas the Monte Carlo runs took 1.25 min.

Appendix E: Discussion of Least-Squares Equation

Equation (1) is the standard "weighted least-squares estimate" equation. One derivation is as follows. Under the assumption that perturbations in the state vector at burnout (i.e., the vector of position and velocity coordinates) are small enough so that linear perturbation theory is valid, the variation in observed values (angles, rates, etc.) can be expressed as

$$\Delta y_i' = y_{i_{\text{measured}}} - y_{i_{\text{nominal}}} = \sum a_{ij}'\Delta x_j = \sum_j \left(\frac{\partial y_i}{\partial x_j}\right) \Delta x_j \quad \text{(E1)}$$

The estimate of the Δx_j is chosen to minimize

$$S = \sum_i \epsilon_i{}^2 = \sum_i \left[\left(\sum_j \frac{a_{ij}'}{\sigma_i}\right)\Delta x_j - \left(\frac{\Delta y_i'}{\sigma_i}\right)\right]^2 = \sum_i \left[\sum_j a_{ij}\Delta x_j - \Delta y_i\right]^2 \quad \text{(E2)}$$

Carrying out the minimization procedure gives

$$\left(\frac{\partial S}{\partial \Delta x_k}\right) = 0 = 2\sum_i\sum_j a_{ik}[a_{ij}\Delta x_j - \Delta y x_i] \quad \text{(E3)}$$

or

$$\sum_i a_{ik}\Delta y_i = \sum_i\sum_j a_{ik}\, a_{ij}\Delta x_j$$

In matrix form, (E3) becomes

$$A^T\Delta\bar{y} = (A^TA)\Delta\bar{x} \quad \text{(E4)}$$

or

$$\Delta\bar{x} = (A^TA)^{-1}A^T\Delta\bar{y} = \text{least-square estimate} \quad \text{(E5)}$$

where

$$A = (a_{ij}) = [(1/\sigma_i)(\partial y_i/\partial x_j)]$$

and

$$\Delta\bar{y} = [(y_{i_{\text{measured}}} - y_{i_{\text{nominal}}})/\sigma_i]$$

If the measured values are uncorrelated, the covariance matrix of $\Delta\bar{x}$ is

$$E[\Delta\bar{x}\ \Delta\bar{x}^T] = (A^TA)^{-1}\,A^TE\times[\Delta\bar{y}\ \Delta\bar{y}^T]\,A(A^T A)^{-1} = (A^TA)^{-1}$$

since Δy_i has unit variance and hence $E[\Delta\bar{y}\ \Delta\bar{y}^T] = I =$ identity matrix. When an a priori covariance matrix of burnout errors is available, the estimated perturbation in the state vector is determined from

$$\Delta\bar{x} = (A^TA + \Lambda_{BO}{}^{-1})^{-1}\{(A^TA)[(A^TA)^{-1}A^T\Delta\bar{y}] + \Lambda_{BO}{}^{-1}\,\Delta\bar{x}_{BO}\} \quad \text{(E6)}$$

It should be noted that this equation just weights the two sources of information by their respective inverse covariance matrices.

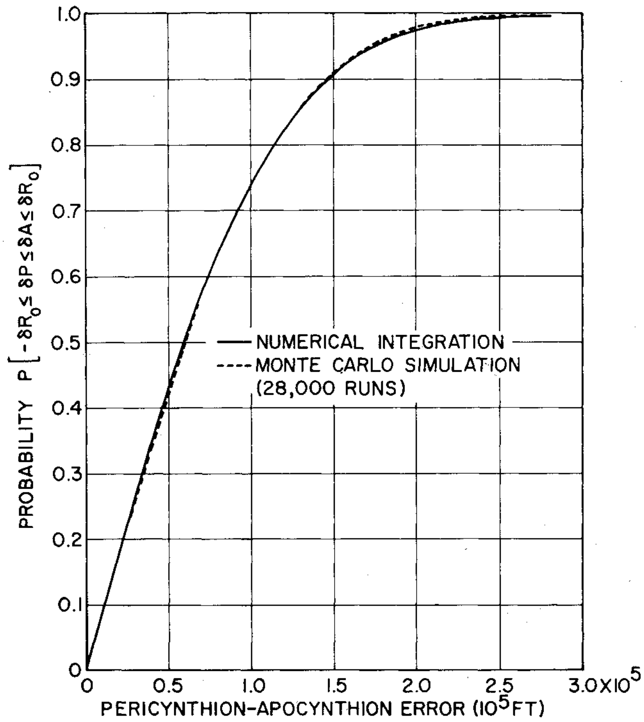

Fig. 12 Cumulative probability of pericynthion-apocynthion error for a lunar parking orbit

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